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$N = 2$ supersymmetry Toda lattice

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Abstract. The vanishing of the 'fermionic-fermionic' component of the strength tensor in the four-dimensional $N = 1$ supersymmetric Yang-Mills theory is used in the definition of the $N = 2$ supersymmetric Toda lattice. The Lax pairs and the Bäcklund transformations for this model are investigated.

1. Introduction

Many papers in the last decade have discussed the powerful mathematical tools which have been developed in completely solving many classical field theoretical models [1, 2]. It was shown that many of the two-dimensional relativistically invariant models admit the so-called Lax pair, the Bäcklund transformation and the Hirota method [3, 4]. These techniques are important in the construction of a special kind of solution, the so-called solitons, and are used in proving the complete integrability of those systems. This framework has been extended to four-dimensional gauge theories [5, 6] as well as to supersymmetric theories [7-10] which however are formal and incomplete.

For example, theoretical physics has developed a new concept of supersymmetry whose main idea is to treat bosons and fermions equally [11]. Mathematically it amounts to incorporating anticommuting variables of Grassman type together with the usual commuting (c-number) variables. It is then natural to ask if the problem of particle-like behaviour in supersymmetric field theories leads to a theory of super-integrable systems. An undoubtedly affirmative answer must be left for future work, but at the very least one should notice that there are two different frameworks of the supersymmetrisation of those relativistic constructions: geometric and algebraic.

In the geometric framework [8, 9] the soliton equations are considered in the form of the Cartan-Maurer equations on the matrix 1-forms belonging to some Lie algebra of a Lie group. Then the supersymmetrisation is performed by generalisation of the Cartan-Maurer equation to the graded supersymmetric Lie algebras. In this way several interesting models have been supersymmetrised. Also the same approach has been applied to extended supersymmetry as, for example, for the $N = 2$ supersymmetric sine-Gordon equation [12] and Liouville equation [13].

The second approach is based on the hidden symmetries of the integrable systems. For example Bogoyavlensky [14] discovered that the classical Toda lattice is connected with the simple Lie algebras. Then Leznov and Saveliev [15, 16] showed that the periodic Toda lattice corresponds to the contragradient Lie algebras. In their approach the vanishing of the strength tensor of the non-Abelian gauge group is utilised as the definition of the Toda lattice. In this way they showed that the non-periodic Toda lattice with free endpoints can be connected with the classical Lie algebras. This connection enables us to investigate this system by means of the inverse scattering transformation (IST). Recently Olshanetsky [17] generalised the two-dimensional Toda lattice to the supersymmetric case. His supersymmetrisation is the supersymmetrisation of the Bogoyavlensky correspondence. He defined the supersymmetric version of the Toda lattice and discovered the connection of this system with the contragradient Lie superalgebras classified by Kac [18].

This paper contains rather formal developments of the concept of the $N=2$ supersymmetric Toda lattice the motivation for which is the same reason that classical Toda lattices have been studied so much recently, i.e. their occurrence in the description of spherically symmetric monopoles and axially symmetric instantons. It is not clear that the classical solution of supersymmetry is in any way relevant but one may think that the $N=1$ and $N=2$ extensions of the Toda lattice should correspond to a certain ansatz for fields of some supersymmetric four-dimensional self-dual Yang-Mills fields which should (by analogy) generalise the axially symmetrical instanton solutions of the ordinary Yang-Mills self-dual fields.

In this paper we show that the $N=2$ supersymmetric Toda lattice is connected with the geometry of the supersymmetric Yang-Mills field theory [11, 19]. We show that the vanishing of the 'fermionic-fermionic' component of the four-dimensional $N=1$ supersymmetric Yang-Mills strength tensor can be used as the definition of the $N=2$ supersymmetric Toda lattice.

This paper is organised as follows. In § 2 we give basic notation used in the theory of supersymmetric Yang-Mills fields and then we formulate our ansatz which reduces the constraint equations to the $N=2$ supersymmetric Toda lattice. In § 3 we describe three different Lax pairs which are connected with the $N=2$ supersymmetric Toda lattice. Section 4 contains the investigations of the Bäcklund transformation for our equation. Here we generalise the Kac-van Moerbecke equation to the $N=2$ supersymmetric case and then we use these equations in the definition of the Bäcklund transformation for the $N=2$ supersymmetric Toda lattice.

2. $N=1$ Yang-Mills superfield and $N=2$ supersymmetric Toda lattice

Let us consider the superspace which is the complexified super-Minkowski space spanned by the spacetime variables x_μ and four anticommuting variables θ_α, $\theta_{\dot\alpha}$. Here $\{\alpha, \dot\alpha\}$ are the usual two-component spinor indices and we assume that $\theta_\alpha^\times = \theta_{\dot\alpha}$ where $\times$ stands for complex conjugation. On this space we represent the algebra of supersymmetry by

$$\mathscr{D}_\alpha = \partial/\partial\theta_\alpha + \mathrm{i}\theta_{\dot\alpha}\,\partial_{\alpha\dot\alpha} \tag{2.1}$$

$$\mathscr{D}_{\dot\alpha} = -\partial/\partial\theta_{\dot\alpha} - \mathrm{i}\theta_\alpha\,\partial_{\alpha\dot\alpha} \tag{2.2}$$

where

$$\partial_{\alpha\dot\alpha} = \sigma^\mu_{\dot\alpha}\,\partial_\mu = \partial/\partial x_{\alpha\dot\alpha} \tag{2.3}$$

$\sigma^\mu = (1, \bar\sigma)$ is a set of Pauli matrices.

This supersymmetric algebra satisfies

$$\{\mathscr{D}_\alpha, \mathscr{D}_\beta\} = 0 = \{\mathscr{D}_{\dot\alpha}, \mathscr{D}_{\dot\beta}\} \tag{2.4}$$

$$\{\mathscr{D}_\alpha, \mathscr{D}_{\dot\beta}\} = -2\mathrm{i}\,\partial_{\alpha\dot\beta}. \tag{2.5}$$

Following the same method as in the Sohnius paper [19] we define the supersymmetric Yang-Mills fields by the supercovariant derivatives

$$\mathscr{D}_\mu = \partial_\mu + \mathrm{i}A_\mu \tag{2.6}$$

$$\mathscr{D}_\alpha = \mathscr{D}_\alpha + \mathrm{i}A_\alpha \tag{2.7}$$

$$\mathscr{D}_{\dot\alpha} = \mathscr{D}_{\dot\alpha} + \mathrm{i}A_{\dot\alpha}. \tag{2.8}$$

Here A_μ, A_α, $A_{\dot\alpha}$ is a Yang-Mills potential and spinor potential respectively and can be considered as the Lie-algebra valued superfield

$$A_\mu = \sum_l T^l A_{\mu l}(x, \theta_\alpha, \theta_{\dot\alpha}) \tag{2.9}$$

$$A_\alpha = \sum_l T^l A_{\alpha l}(x, \theta_\alpha, \theta_{\dot\alpha}) \tag{2.10}$$

$$A_{\dot\alpha} = \sum_l T^l A_{\dot\alpha l}(x, \theta_\alpha, \theta_{\dot\alpha}) \tag{2.11}$$

where T^l are the gauge group generators.

The commutator of two supercovariant derivatives yield the six Yang-Mills field strengths

$$\{\mathscr{D}_\alpha, \mathscr{D}_\beta\} = \mathrm{i}F \qquad \{\mathscr{D}_{\dot\alpha}, \mathscr{D}_{\dot\beta}\} = \mathrm{i}F_{\dot\alpha\dot\beta} \tag{2.12}$$

$$\{\mathscr{D}_\alpha, \mathscr{D}_{\dot\beta}\} = \mathrm{i}F_{\alpha\dot\beta} - 2\mathrm{i}\mathscr{D}_{\alpha\dot\beta} \tag{2.13}$$

$$[\mathscr{D}_\mu, \mathscr{D}_\alpha] = \mathrm{i}F_{\mu\alpha} \qquad [\mathscr{D}_\mu, \mathscr{D}_{\dot\alpha}] = \mathrm{i}F_{\mu\dot\alpha} \tag{2.14}$$

$$[\mathscr{D}_\mu, \mathscr{D}_\nu] = \mathrm{i}F_{\mu\nu} \tag{2.15}$$

where $\{A, B\} = AB + BA$ and $[A, B] = AB - BA$.

Each tensor component of F represents a full superfield multiplet. Most of these components are superfluous and are usually eliminated by the constraint equations. For $N=1$ the constraints are

$$F_{\alpha\beta} = F_{\dot\alpha\dot\beta} = F_{\alpha\dot\beta} = 0 \tag{2.16}$$

which does not provide us with a flat theory.

The following formula

$$A_\alpha = h^{-1}\mathscr{D}_\alpha h \tag{2.17}$$

where h is an arbitrary superfield, is usually considered as the solution of

$$F_{11} = 0 = F_{22}. \tag{2.18}$$

However, there are other possibilities for the solution of (2.18):

$$A_1 = f_1 b_1 = b_1 f_1 \tag{2.19}$$

$$A_2 = f_2 b_2 = b_2 f_2 \tag{2.20}$$

where f_α and b_α are an arbitrary fermionic $f_1^2 = f_2^2 = 0$ and bosonic superfield respectively such that

$$\mathscr{D}_1 b_1 = 0 = \mathscr{D}_2 b_2 \tag{2.21}$$

$$\mathscr{D}_1 f_1 = 0 = \mathscr{D}_2 f_2. \tag{2.22}$$

Assuming that we have the solution of (2.18) in the form of (2.19) and (2.20) then the constraint $F_{12}=0$ gives us the equation on the functions f_α and b_α which in the following we will consider to be the equation of motion for the supersymmetric Toda lattice. Indeed let us consider the SU(2) case for which we have the following ansatz:

$$A_1 = f_1 f_2 \mathscr{D}_1 \ln \phi \cdot H + 2 f_1 \phi^\times E^+ \tag{2.23}$$

$$A_2 = f_2 f_1 \mathscr{D}_2 \ln \phi \cdot H + 2 f_2 E^- \tag{2.24}$$

where $H=\sigma_3$, $E^+=\frac{1}{2}(\sigma_1+\mathrm{i}\sigma_2)$, $E^-=\frac{1}{2}(\sigma_1-\mathrm{i}\sigma_2)$ are the generators of the SU(2). f_1 and f_2 are arbitrary four-dimensional fermionic chiral superfields, e.g. $\mathscr{D}_\alpha f_\beta=0$ while $\phi=\phi(x_{1\dot{1}}, x_{2\dot{2}}, \theta_\alpha, \theta_{\dot{\alpha}})$ is an arbitrary bosonic superfield such that

$$\mathscr{D}_1 \phi^\times = 0 = \mathscr{D}_2 \phi^\times \tag{2.25}$$

where $\times$ denotes complex conjugation.

Introducing (2.23) and (2.24) to $F_{12}=0$ we find that

$$\mathscr{D}_2 \mathscr{D}_1 \ln \phi = -2\mathrm{i}\phi^\times \tag{2.26}$$

constitutes the $N=2$ supersymmetric Liouville equation and coincides with the equation obtained by Ivanov and Krivonos [13].

Notice that the assumption that ϕ is two-dimensional in Minkowski space and four-dimensional in Grassman space is equivalent to the statement that our Liouville equation is the $N=2$ supersymmetric equation in two-dimensional spacetime. The conditions (2.25) are the Grassman [20] analyticity conditions. They reduce the complex $N=2$ superfield ϕ to a complex $N=1$ superfield by

$$\phi = \phi(x_{1\dot{1}} - \mathrm{i}\theta_{\dot{1}}\theta_1, x_{2\dot{2}} - \mathrm{i}\theta_{\dot{2}}\theta_2, \theta_1, \theta_2). \tag{2.27}$$

Equation (2.26) can be obtained from the superfield action of the form (with the solved conditions (2.25))

$$S = \int \mathrm{d}^2x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, (\phi^\times \mathscr{D}_2 \mathscr{D}_1 \phi - 2\mathrm{i} \exp \phi^\times) + \mathrm{HC} \tag{2.28}$$

where $\Phi = \exp \phi$.

Now the generalisation to the Toda lattice is straightforward. Notice that in the expansion (2.23) and (2.24) the generators are the linear combinations of the generators in the Cartan–Weyl basis and therefore constitute the Chevalley basis [21]. Hence for the larger group we can put

$$A_1 = f_1 f_2 \mathscr{D}_1 \ln \phi_i H_i + 2 f_1 K_{ij} \phi_j^\times E_i^+ \tag{2.29}$$

$$A_2 = f_2 f_1 \mathscr{D}_2 \ln \phi_i H_i + 2 f_2 \sum_i E_i^- \tag{2.30}$$

where we have the following relations:

$$[H_i, H_j] = 0 \tag{2.31}$$

$$[H_i, E_j^\pm] = \pm K_{ji} E_j^\pm \tag{2.32}$$

$$[E_i^+, E_j^-] = \delta_{ij} H_j \tag{2.33}$$

and $(K_{ji}) = K$ are the elements of the Cartan matrix. Here the superfields f_1 and f_2 are the same as before and

$$\mathscr{D}_1 \phi_i^\times = 0 = \mathscr{D}_2 \phi_i^\times. \tag{2.34}$$

With these relations the condition $F_{12}=0$ gives us

$$\mathscr{D}_2\mathscr{D}_1 \ln \phi_i H_i + 2\mathrm{i} K_{ij}\phi_j^{\times} H_i = 0. \tag{2.35}$$

Introducing $\phi_i = \exp V_i$ and $V_i = K_{ij}\phi_j$ equation (2.35) reduces to

$$\mathscr{D}_2\mathscr{D}_1\phi_j = -2\mathrm{i} \exp K_{js}\phi_s^{\times}. \tag{2.36}$$

This is our desired $N=2$ supersymmetric Toda lattice.

Equation (2.36) can be obtained from the following supersymmetric action (with the solved conditions (2.34))

$$S = \int \mathrm{d}^2x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2\left(\sum_j \phi_j^{\times}\mathscr{D}_2\mathscr{D}_{1j} - U(\phi^{\times})\right) + \text{HC} \tag{2.37}$$

$$U(\phi^{\times}) = \sum_j \frac{1}{(\beta_j\beta_j)} \exp 2(\phi^{\times}\beta_j) \tag{2.38}$$

where $\{\beta_j\}$ is a set of simple roots connected with a given gauge group. Indeed if we put

$$\Phi_j = \sum_s \beta_s^j \phi_s/(\beta_s\beta_s) \tag{2.39}$$

and we use the definition of the Cartan matrix

$$K_{ij} = 2(\beta_i\beta_j)/(\beta_j\beta_j) \tag{2.40}$$

we easily recover equation (2.36).

3. The associated linear problem

According to the IST it is necessary to represent the equation (2.36) as an integrability condition for the system of linear equations

$$\mathscr{D}_1\psi = -\mathrm{i}\, U\psi \tag{3.1}$$

$$\mathscr{D}_2\psi = -\mathrm{i}\, V\psi \tag{3.2}$$

where U and V are some operators which we want to define. Now the compatibility condition takes the form

$$\mathscr{D}_2 U + \mathscr{D}_1 V + \mathrm{i}\{UV\} = 0. \tag{3.3}$$

We supplement this condition with

$$\mathscr{D}_1\mathscr{D}_1\psi = 0 = \mathscr{D}_1 U + \mathrm{i}\, UU \tag{3.4}$$

$$\mathscr{D}_2\mathscr{D}_2\psi = 0 = \mathscr{D}_1 V + \mathrm{i}\, VV. \tag{3.5}$$

Now one can easily check that the equation $F_{12}=0$ is equivalent to the condition (3.3) if we choose the following representation of U and V:

$$\mathscr{D}_1\psi = -\mathrm{i}A_1\psi \tag{3.6}$$

$$\mathscr{D}_2\psi = -\mathrm{i}A_2\psi \tag{3.7}$$

where A_1 and A_2 are the spinor superpotentials of the Yang–Mills fields.

However notice that in our IST there is no spectral parameter. We can introduce it in three different ways.

(1)

$$\mathscr{D}_1\psi = -\frac{\mathrm{i}A_1}{1-\lambda}\psi \tag{3.8}$$

$$\mathscr{D}_2\psi = -\frac{\mathrm{i}A_2}{1+\lambda}\psi. \tag{3.9}$$

For this IST compatibility conditions (3.3)-(3.5) reduce to

$$\mathscr{D}_2 A_1 - \mathscr{D}_1 A_1 = 0 \tag{3.10}$$

$$F_{12} = 0 \tag{3.11}$$

$$A_1A_1 = 0 = A_2A_2, \qquad \mathscr{D}_1 A_1 = 0 = \mathscr{D}_2 A_2. \tag{3.12}$$

These conditions restrict our choice of A_α but our potentials defined in the previous section satisfy (3.10)-(3.12). Notice that this IST is similar to the IST for the two-dimensional chiral models considered by Mikhailov and Zakharov [22]. There is a basic difference that in the chiral models the analogous equation to (3.10) plays the role of the equation of motion while the zero curvature condition (3.11) is trivially satisfied. Here the situation is quite the reverse. This correspondence also exists for the normal Yang-Mills field equation. It was shown by the present author [23] that the non-Abelian two-dimensional Toda lattice can be considered as the model of discrete chiral fields and in the continuum case it reduces to the three-dimensional self-dual sector of the Yang-Mills field.

(2) Let us put

$$U = U_0 + \lambda U_1 \tag{3.13}$$

$$V = \lambda^{-1} V_0 \tag{3.14}$$

where

$$U_0 = 2f_1 f_2 \mathscr{D}_1 \ln \phi_i H_i \tag{3.15}$$

$$U_1 = 2f_1 K_{ij} \phi_j^{\times} E_i^{+} \tag{3.16}$$

$$V_0 = 2f_2 \sum_i E_i^{-} \tag{3.17}$$

where the superfields f_1, f_2, ϕ and the generators H_i, $E_i^{\pm}$ satisfy the same conditions as in the previous case.

This IST is the direct generalisation of IST from the $N=1$ supersymmetric Toda lattice considered by Olshanetsky [17]. The condition (3.3) gives us the equation (2.36) while (3.4) and (3.5) reduce to the equation (2.34), i.e. give us the Grassmannian analyticity of ϕ_i.

(3) Let us consider the IST in the following form:

$$(\mathscr{D}_1 + \lambda\mathscr{D}_2)\psi = -\mathrm{i}(A_1 + \lambda A_2)\psi \tag{3.18}$$

which is the IST considered by Volovich [24] for the supersymmetric Yang-Mills fields.

In this case we have only the following condition of compatibility:

$$(\mathscr{D}_1+\lambda\mathscr{D}_2)(\mathscr{D}_1+\lambda\mathscr{D}_2)\psi=0. \tag{3.19}$$

Notice that our ansatz for the supersymmetric Toda lattice satisfies this IST.

4. The Bäcklund transformation for the $N=2$ supersymmetric SU(N) Toda lattice

For the non-periodic SU(N) Toda lattice there exists an auto-Bäcklund transformation. This transformation is the generalisation of the so-called Kac–Van Moerbeke equations to two-dimensional spacetime [16]. Here we use the $N=2$ supersymmetric version of the Kac–van Moerbeke equation to construct the auto-Bäcklund transformation for the $N=2$ supersymmetric Toda lattice. First let us construct the $N=2$ supersymmetric Kac–van Moerbeke equations. They can be written down as

$$\mathscr{D}_1 \ln N_{2\alpha-1}=-f_{2\alpha-1}N^{\times}_{2\alpha}+f_{2\alpha-3}N^{\times}_{2\alpha-2} \tag{4.1}$$

$$\mathscr{D}_2 \ln N_{2\alpha}=f_{2\alpha}N^{\times}_{2\alpha+1}-f_{2\alpha-2}N^{\times}_{2\alpha-1} \tag{4.2}$$

$$1<a<r+1, \qquad N_0=0=N_{2r+2}. \tag{4.3}$$

Here N_a, $1<a<2r+1$ are arbitrary bosonic superfields such that

$$\mathscr{D}_1 N^{\times}_a=\mathscr{D}_2 N^{\times}_a=0 \tag{4.4}$$

while f_a are fermionic superfields such that

$$\tfrac{1}{2}\mathrm{i}\mathscr{D}_2 f_{2\alpha-1}=N^{\times}_{2\alpha+1}-N^{\times}_{2\alpha-1} \tag{4.5}$$

$$\tfrac{1}{2}\mathrm{i}\mathscr{D}_1 f_{2\alpha}=N^{\times}_{2\alpha+2}-N^{\times}_{2\alpha}. \tag{4.6}$$

Moreover from the condition $\mathscr{D}_1\mathscr{D}_1=0=\mathscr{D}_2\mathscr{D}_2$ it follows that

$$\mathscr{D}_1 f_{2\alpha-1}=0=\mathscr{D}_2 f_{2\alpha}. \tag{4.7}$$

Now, as one can easily verify, the following functions

$$\Phi_\alpha=\ln(N_{2\alpha-1}N_{2\alpha}), \qquad \Phi'_\alpha=\ln(N_{2\alpha}N_{2\alpha+1}) \tag{4.8}$$

satisfy the following equations:

$$\tfrac{1}{2}\mathrm{i}\mathscr{D}_2\mathscr{D}_1\Phi_\alpha=2\exp\phi^{\times}_\alpha-\exp\phi^{\times}_{\alpha+1}-\exp\phi^{\times}_{\alpha-1} \tag{4.9}$$

$$\tfrac{1}{2}\mathrm{i}\mathscr{D}_2\mathscr{D}_1\phi'_\alpha=2\exp\phi'^{\times}_\alpha-\exp\phi'^{\times}_{\alpha+1}-\exp\phi'_{\alpha-1} \tag{4.10}$$

$$\Phi_0=\Phi_{r+1}=-\infty=\Phi'_0=\Phi'_{r+1}.$$

Now transforming Φ_α to $\Phi_\alpha=K_{\alpha\beta}\phi_\beta$ where

$$K_{\alpha\beta}=\begin{pmatrix} 2\delta_{\alpha\beta} \\ -1\delta_{\alpha,\beta\pm1}\end{pmatrix} \tag{4.11}$$

is the Cartan matrix for the SU(N) group, we obtain equation (2.36). From these considerations we see that our Bäcklund transformation relates the solutions of some $N=2$ supersymmetric Toda lattice with the second Toda lattice. Notice that for the SU(2) case our Bäcklund transformation reduces to the Bäcklund transformation for the $N=2$ supersymmetric Liouville equation. However for these equations it is possible to define additionally two different Bäcklund transformations. The first is the non-auto

Bäcklund transformation which relates the $N=2$ supersymmetric Liouville equation with the $N=2$ supersymmetric d'Alembert. This can be written down as

$$\mathscr{D}_2(\phi+h)=\lambda\chi \exp\tfrac{1}{2}(\phi^\times - h^\times) \tag{4.12}$$

$$\mathscr{D}_1(\phi-h)=\lambda\chi^\times \exp\tfrac{1}{2}(\phi^\times + h^\times) \tag{4.13}$$

$$\mathscr{D}_1\chi=(\mathrm{i}/\lambda) \exp\tfrac{1}{2}(\phi^\times + h^\times) \tag{4.14}$$

$$\mathscr{D}_2\chi^\times=-(\mathrm{i}/\lambda) \exp\tfrac{1}{2}(\phi^\times - h^\times) \tag{4.15}$$

$$\mathscr{D}_2\chi=0=\mathscr{D}_1\chi^\times \tag{4.16}$$

$$\mathscr{D}_2\phi^\times=\mathscr{D}_2h^\times=\mathscr{D}_1\phi^\times=\mathscr{D}_1h^\times=0 \tag{4.17}$$

where λ is an arbitrary parameter.

Now the integrability conditions for our Bäcklund transformation give us the $N=2$ supersymmetric Liouville equation (2.33) while for the function h we obtain

$$\mathscr{D}_1\mathscr{D}_2h=0 \tag{4.18}$$

i.e. the $N=2$ supersymmetric d'Alembert equation. We can easily solve equation (4.18) with (4.17) which gives us

$$h=h_1+h_2=h_1(2\dot{2}-\mathrm{i}\theta_2\theta_2,\ \theta_1)+h_2(1\dot{1}-\mathrm{i}\theta_1\theta_1,\ \theta_1). \tag{4.19}$$

Let us present the second auto-Bäcklund transformation for the $N=2$ supersymmetric Liouville equation obtained by Ivanov and Krivonos [25]. It has the following form:

$$\mathscr{D}_2(\phi_1+\phi_2)=(1/\lambda)f\cosh\tfrac{1}{2}(\phi_1^\times-\phi_2^\times) \tag{4.20}$$

$$\mathscr{D}_1(\phi_1-\phi_2)=\lambda f^\times \exp\tfrac{1}{2}(\phi_1^\times+\phi_2^\times) \tag{4.21}$$

$$\mathscr{D}_1f^\times=0=\mathscr{D}_2f \tag{4.22}$$

$$\mathscr{D}_2\phi_1^\times=0=\mathscr{D}_2\phi_2^\times \tag{4.23}$$

$$\mathscr{D}_1\phi_1^\times=0=d_1\phi_2^\times \tag{4.24}$$

$$\begin{aligned}&\mathscr{D}_1f=2\lambda\exp\tfrac{1}{2}(\phi_1^\times+\phi_2^\times),\\ &\mathscr{D}_2f^\times=-(2/\lambda)\sinh\tfrac{1}{2}(\phi_1^\times-\phi_2^\times).\end{aligned} \tag{4.25}$$

Finally let us discuss the reduction of our Bäcklund transformations to the $N=1$ and $N=0$ supersymmetric cases. The reduction from $N=2$ to $N=1$ can be formally achieved by putting $\theta_{\dot{1}}=\theta_{\dot{2}}=0$ and assuming that $\theta_1\theta_2$, $\phi_1\phi_2$ are real. Using this prescription our Bäcklund transformations (4.12)-(4.17) and (4.20)-(4.25) reduce to those given in [7] while the transformation (4.1)-(4.6) reduces to the Bäcklund transformation for the $N=1$ supersymmetric SU(N) Toda lattice. The reduction from $N=1$ to $N=0$ can be formally achieved by putting $\theta_1=\theta_2=0$. Then our Bäcklund transformations reduces to the well known Bäcklund transformation.

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